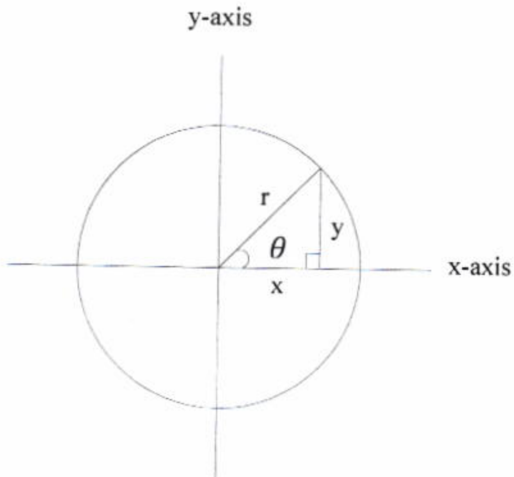
Trigonometric Functions: الدوال المثلثيه

Given a circle of radius (r) and P(x, y) is any point of the circle then:

- Sine of θ ⇒ $\sin\theta = \frac{y}{r}$
- Cosine of θ ⇒ $\cos\theta = \frac{x}{r}$
- Tangent of θ ⇒ $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$
- Secant of θ ⇒ $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$
- Cosecant of θ ⇒ $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y}$
- Cotangent of θ ⇒ $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$



From the above triangle we can say:

$r^2 = x^2 + y^2$ (فيثاغورس)

But: $x = r\cos\theta$ & $y = r\sin\theta$

$\therefore r^2 = (r\cos\theta)^2 + (r\sin\theta)^2$

$r^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$r^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\sin^2\theta + \cos^2\theta = 1$

Some of Trigonometric Identities:

- $\sin^2 x + \cos^2 x = 1$ ……… (1)
- $\tan^2 x + 1 = \sec^2 x$ (ناتجة من قسمة المعادله (1) على $\cos^2 x$)
- $1 + \cot^2 x = \csc^2 x$ (ناتجة من قسمة المعادله (1) على $\sin^2 x$)
- $\sin 2x = 2\sin x\cos x$
- $\cos 2x = \cos^2 x - \sin^2 x$
- $\because \cos 2x = \cos^2 x - \sin^2 x$

  $\cos 2x = (1 - \sin^2 x) - \sin^2 x$

  $\cos 2x = 1 - 2\sin^2 x$

  $2\sin^2 x = 1 - \cos 2x \Rightarrow \therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$

Prove that: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

Right side $= \frac{1}{2}(1-\cos 2x)$

$= \frac{1}{2}[1-(\cos^2 x - \sin^2 x)]$

$= \frac{1}{2}[1-(1-\sin^2 x - \sin^2 x)$

$= \frac{1}{2}[1-1+2\sin^2 x]$

$= \sin^2 x =$ left side

- $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ ------------------ Prove that
- $\sin(x+y) = \sin x \cos x + \cos x \sin y$
- $\sin(x-y) = \sin x \cos x - \cos x \sin y$
- $\cos(x+y) = \cos x \cos y - \sin x \sin y$
- $\cos(x-y) = \cos x \cos y + \sin x \sin y$
- $\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x \tan y}$
- $\tan(x-y) = \frac{\tan x - \tan y}{1+\tan x \tan y}$
- $\sin A.\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$
- $\cos A.\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$
- $\sin A.\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$

- قيمة الـ π مع الزوايا تساوي (180) بينما مع الأرقام تساوي (3.14)
- اشارة الدوال المثلثيه: كل جار ظالم جتا مصيبه

الربع الثاني (sin) +	الربع الأول (sin, cos, tan) +
الربع الثالث (tan) +	الربع الرابع (cos) +

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❖ اضافة أو طرح أي عدد من الدورات لايغير من قيمة الزاويه (2π دوره واحده & 4π دورتان & 6π ثلاث دورات وهكذا....).

❖ $90 = \frac{\pi}{2}$ & $180 = \pi$ & $270 = \frac{3\pi}{2}$ & $360 = 2\pi$

➢ الزوايا المكمله (π & 2π): موقع الربع يحدد الأشاره وتبقى الداله نفسها.

الربع الثالث:
$$\sin(\pi+\theta) = -\sin\theta$$
$$\cos(\pi+\theta) = -\cos\theta$$
$$\tan(\pi+\theta) = +\tan\theta$$

الربع الثاني:
$$\sin(\pi-\theta) = +\sin\theta$$
$$\cos(\pi-\theta) = -\cos\theta$$
$$\tan(\pi-\theta) = -\tan\theta$$

الربع الرابع:
$$\sin(-\theta) = -\sin$$
$$\cos(-\theta) = +\cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

الربع الرابع:
$$\sin(2\pi-\theta) = -\sin\theta$$
$$\cos(2\pi-\theta) = +\cos\theta$$
$$\tan(2\pi-\theta) = -\tan\theta$$

➢ الزوايا المتممه ($\frac{\pi}{2}$ & $\frac{3\pi}{2}$): موقع الربع يحدد الأشاره وتقلب الداله الى متممتها. مثلاً ال sin في الربع الأول تكون موجبه ثم تقلب الى cos .

الربع الثاني:
$$\sin(\frac{\pi}{2}+\theta) = +\cos\theta$$
$$\cos(\frac{\pi}{2}+\theta) = -\sin\theta$$
$$\tan(\frac{\pi}{2}+\theta) = -\cot\theta$$

الربع الأول:
$$\sin(\frac{\pi}{2}-\theta) = +\cos\theta$$
$$\cos(\frac{\pi}{2}-\theta) = +\sin\theta$$
$$\tan(\frac{\pi}{2}-\theta) = +\cot\theta$$

الربع الرابع:
$$\sin(\frac{3\pi}{2}+\theta) = -\cos\theta$$
$$\cos(\frac{3\pi}{2}+\theta) = +\sin\theta$$
$$\tan(\frac{3\pi}{2}+\theta) = -\cot\theta$$

الربع الثالث:
$$\sin(\frac{3\pi}{2}-\theta) = -\cos\theta$$
$$\cos(\frac{3\pi}{2}-\theta) = -\sin\theta$$
$$\tan(\frac{3\pi}{2}-\theta) = +\cot\theta$$

Example 1: **sin10x**

$= 2\sin 5x \cos 5x$

Example 2: **cos4x**

$= \cos^2 2x - \sin^2 sx$

Example 3: $\mathbf{\cos^2 7x}$

$= \frac{1}{2}(1 + \cos 14x)$

Example 4: $\mathbf{\sin^2 15}$

$= \frac{1}{2}(1 - \cos 30)$

Example 5: **cos330**

$= \cos(270 + 60)$

$= + \sin 60$

Example 6: **cos330**

$= \cos(360 - 30)$

$= + \cos 30$

Example 7: **sin150**

$= \sin(180 - 30)$

$= + \sin 30$

Functions: الدوال

Introduction: المقدمة

Rules for Inequalities:

If a, b, and c are real numbers, then:

1) $a < b \rightarrow a + c < b + c$
2) $a < b \rightarrow a - c < b - c$
3) $a < b \text{ and } c > 0 \rightarrow ac < bc$
4) $a < b \text{ and } c < 0 \rightarrow bc < ac$

Special case: $a < b \rightarrow -b < -a$

5) $a > 0 \rightarrow \frac{1}{a} > 0$
6) If a and b are both positive or both negative, then $a < b \rightarrow \frac{1}{b} < \frac{1}{a}$

Example: Solve the following:

a) $2x - 1 < x + 3$

$2x - x < 1 + 3$

$x < 4$

b) $-\frac{x}{3} < 2x + 1$

$-x < 6x + 3$

$0 < 7x + 3$

$-\frac{3}{7} < x$

Absolute Value Properties

1) $|-a| = |a|$
2) $|ab| = |a||b|$
3) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
4) $|a + b| \leq |a| + |b|$

Absolute Values and Intervals

If a is any positive number, then:

1) $|x| = a \quad if\ and\ only\ if\ x = \pm a$
2) $|x| < a \quad if\ and\ only\ if -a < x < a$
3) $|x| \leq a \quad if\ and\ only\ if -a \leq x \leq a$
4) $|x| > a \quad if\ and\ only\ if\ x > a\ or\ x < -a$
5) $|x| \geq a \quad if\ and\ only\ if\ x \geq a\ or\ x \leq -a$

Example: Solve the following:

a) $|2x - 3| = 7$

$2x - 3 = 7 \qquad\qquad 2x - 3 = -7$

$2x = 10 \qquad\qquad 2x = -4$

$x = 5 \qquad\qquad x = -2$

$The\ solution\ is\ x = 5\ and\ x = -2$

b) $|5 - \frac{2}{x}| < 1$

$-1 < 5 - \frac{2}{x} < 1$

$-6 < -\frac{2}{x} < -4 \qquad\qquad (multiply\ by -\frac{1}{2})$

$3 > \frac{1}{x} > 2 \quad \rightarrow \quad \therefore\ \frac{1}{3} < x < \frac{1}{2}$

c) $|2x - 3| \leq 1$

$-1 \leq 2x - 3 \leq 1$

$2 \leq 2x \leq 4 \qquad\qquad (divide\ by\ 2)$

$1 \leq x \leq 2$

d) $|2x - 3| \geq 1$

$2x - 3 \geq 1 \quad or \quad 2x - 3 \leq -1$

$2x \geq 4 \quad or \quad 2x \leq 2$

$x \geq 2 \quad or \quad x \leq 1$

$The\ solution\ is\ (-\infty, 1] \cup [2, \infty)$

Functions: الدوال

The Domains and Ranges:

Domain: هو مجال القيم الحقيقيه لـ (x) التي تأخذ قيم حقيقيه لـ (y).

وصيغة المعادله تكون: Equation $\Rightarrow y = f(x)$

Range: هو مجال القيم الحقيقيه لـ (y) التي تأخذ قيم حقيقيه لـ (x).

وصيغة المعادله تكون: Equation $\Rightarrow x = f(y)$

Domain and Range are divided to Bounded & Infinite Intervals:

	Bounded Intervals الفترات المحدده		Infinite Intervals الفترات الغير محدده
1	$a < x < b$ a (——) b $\Rightarrow$ open interval $D_f, R_f : a < x < b$ or (a, b)	1	$-\infty < x < \infty$ $-\infty$ ←—0—→ ∞ $D_f, R_f : -\infty < x < \infty$ or $(-\infty, \infty)$ or R (all real numbers)
2	$a \leq x \leq b$ a [——] b $\Rightarrow$ close interval $D_f, R_f : a \leq x \leq b$ or $[a, b]$	2	$a < x$ a (——→ ∞ $D_f, R_f : x > a$ or (a, ∞)
3	$a \leq x < b$ a [——) b $\Rightarrow$ half-open interval $D_f, R_f : a \leq x < b$ or $[a, b)$	3	$a \leq x$ a [——→ ∞ $D_f, R_f : x \geq a$ or $[a, \infty)$
4	$a < x \leq b$ a (——] b $\Rightarrow$ half-open interval $D_f, R_f : a < x \leq b$ or $(a, b]$	4	$x < b$ $-\infty$ ←——) b $D_f, R_f : x < b$ or $(-\infty, b)$
		5	$x \leq b$ $-\infty$ ←——] b $D_f, R_f : x \leq b$ or $(-\infty, b]$

Example 1: $f(x) = x + 15$ $\quad\quad f(x) = y$

D_F: R

$y = x + 15 \Rightarrow x = y - 15$

R_F: R

Example 2: $y = \frac{3}{x-2}$ $\quad\quad$ المقام $\neq 0$

الشرط $\Rightarrow x - 2 \neq 0 \Rightarrow x \neq 2$

D_F: R / {2}

$y = \dfrac{3}{x-2} \Rightarrow yx - 2y = 3$

$yx = 3 + 2y \Rightarrow x = \dfrac{3+2y}{y}$

R_F: R / {0}

Example 3: $y = \sqrt{x-1}$ $\quad\quad$ لايجوز أن تكون القيمة سالبة تحت الجذر (قيمه خيالية)

الشرط $\Rightarrow x - 1 \geq 0 \Rightarrow x \geq 1$

D_F: x: $x \geq 1$ or $[1,\infty)$

$y = \sqrt{x-1} \Rightarrow y^2 = x - 1$

$x = y^2 + 1$

R_F: y: $y \geq 0$ or $[0,\infty)$ نأخذ القيم الموجبة والصفر فقط لأن الدالة الأصلية دالة جذرية

Example 4: $y = \frac{1}{\sqrt{x-1}}$

الشرط $\Rightarrow x - 1 > 0 \Rightarrow x > 1$

D_F: x: $x > 1$ or $(1,\infty)$

$y = \dfrac{1}{\sqrt{x-1}} \Rightarrow y^2 = \dfrac{1}{x-1}$

$y^2x - y^2 = 1 \Rightarrow x = \dfrac{1-y^2}{y^2} \Rightarrow x = 1 + \dfrac{1}{y^2}$

R_F: y: $y > 0$ نأخذ القيم الموجبة فقط لأن الدالة الأصلية دالة جذرية وكسرية

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Example 5: $y = \sqrt{1 - x^2}$

الشرط $\Rightarrow 1 - x^2 \geq 0$

$(1 - x)(1 + x) \geq 0$

---- -1[++++] 1----

من رسم الدالة D_F: $-1 \leq x \leq 1$ or [–1, 1]

$y = \sqrt{1 - x^2} \Rightarrow y^2 = 1 - x^2$

$x^2 = 1 - y^2 \Rightarrow x = \sqrt{1 - y^2}$

الشرط $\Rightarrow 1 - y^2 \geq 0$

$(1 - y)(1 + y) \geq 0$

من رسم الدالة R_F: $0 \leq y \leq 1$ or [0, 1]

Example 6: $y = x + \frac{1}{x}$

الشرط $\Rightarrow x \neq 0$

D_F: R / {0}

$y = x + \frac{1}{x} \Rightarrow y = \frac{x^2 + 1}{x} \Rightarrow yx = x^2 + 1$

$x^2 - yx + 1 = 0$

المعادله العامه $\Rightarrow ax^2 + bx + c = 0$

$x = \frac{y \pm \sqrt{y^2 - (4*1*1)}}{2*1}$

قانون الدستور $\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

الشرط $\Rightarrow y^2 - 4 \geq 0$

$a = 1$ & $b = -y$ & $c = 1$

$(y + 2)(y - 2) \geq 0$

++++ -2[----] 2++++

من رسم الدالة R_F: $y \geq 2 \cup y \leq -2$ or R/ (–2, 2)

Example 7: $y = \sqrt{x^2 - 3x}$

الشرط $\Rightarrow x^2 - 3x \geq 0$

$x(x - 3) \geq 0$

من رسم الدالة D_F: $x \geq 3 \cup x \leq 0$ or R/ (0, 3)

++++ 0[----]3++++

$y = \sqrt{x^2 - 3x} \Rightarrow y^2 = x^2 - 3x$

$x^2 - 3x - y^2 = 0$

$a = 1$ & $b = -3$ & $c = -y^2$

$$x = \frac{3 \pm \sqrt{9 - [4 * 1 * (-1)]}}{2 * 1}$$

$9 + 4y^2 \geq 0$

من رسم الدالة $R_{F:}$ $y \geq 0$ or $[0, \infty)$

Note: If f(x) = f & g(x) = g then:

Domain of $f + g$ & $f - g$ & $f * g = D_F \cap D_g$

But the domain of $f/g = D_F \cap D_g / g(x) = 0$

Example 8: Find the domain only for $y = \sqrt{x-3} + \sqrt{3-2x}$

$\sqrt{x-3} \quad + \quad \sqrt{3-2x}$

$\Downarrow \qquad \Downarrow$

$x - 3 \geq 0 \qquad 3 - 2x \geq 0$

$x \geq 3 \qquad \frac{3}{2} \geq 0$

$D_F = Ø$ كمية خالية

+++1.5] ----- [3+++

(1) ∩ (2) = Ø

Example 9: Find the domain only for $y = \sqrt{4-x} + \sqrt{x-1}$

$\sqrt{4-x} \quad + \quad \sqrt{x-1}$

$\Downarrow \qquad \Downarrow$

$4 - x \geq 0 \qquad x - 1 \geq 0$

$4 \geq x \qquad x \geq 1$

$D_F = [1, 4]$

1[ ]4

(1) ∩ (2) = [1, 4]

Example 10: Find the domain only for $y = \frac{\sqrt{x-1}}{\sqrt{4-x}}$

$\frac{\sqrt{x-1}}{\sqrt{4-x}}$ ↗ $x - 1 \geq 0 \Rightarrow x \geq 1$

$\frac{\sqrt{x-1}}{\sqrt{4-x}}$ ↘ $4 - x > 0 \Rightarrow 4 > x$

1[ )4

(1) ∩ (2) = [1, 4)

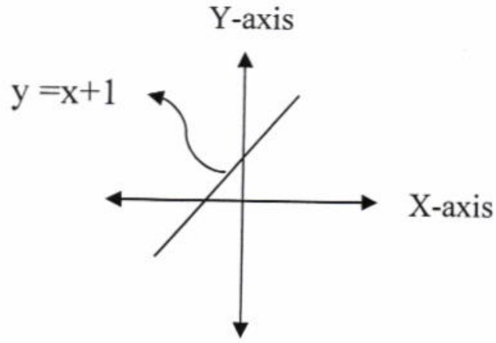
$D_F = [1, 4)$

Note: The projection of the graph of a function (f) on the x-axis is (D_F) and on the y-axis is (R_F).

Example 11: $y = x+1$

D_F: R

R_F: R

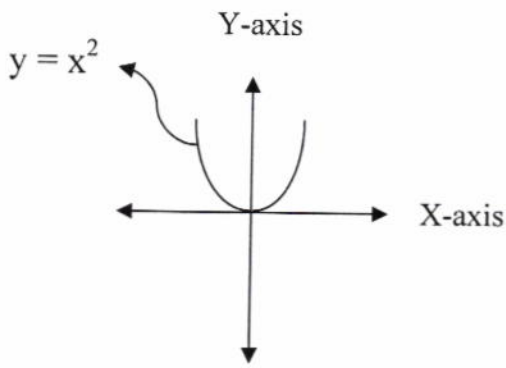


Example 12: $y = x^2$

D_F: R

$x = \sqrt{y}$

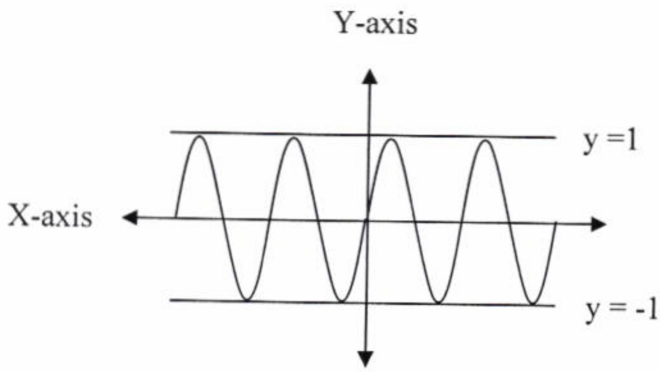
R_F: y: $y \geq 0$



Example 13: $y = \text{Sinx}$

D_F: R

$R_F = [-1, 1]$



Example 14: $y = 2\text{Sinx}$

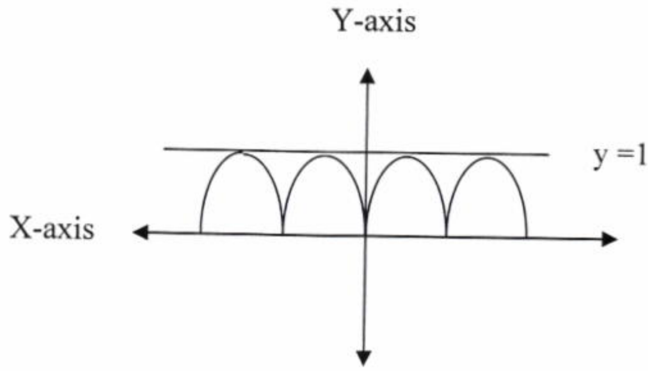
D_F: R

$R_F = [-2, 2]$ نفس الرسم السابق ولكن مضاعف

Example 15: $y = 2 + 3\sin x$

D_F: R

$R_F = [-1, 5]$



Example 16: $y = \sin^2 x$

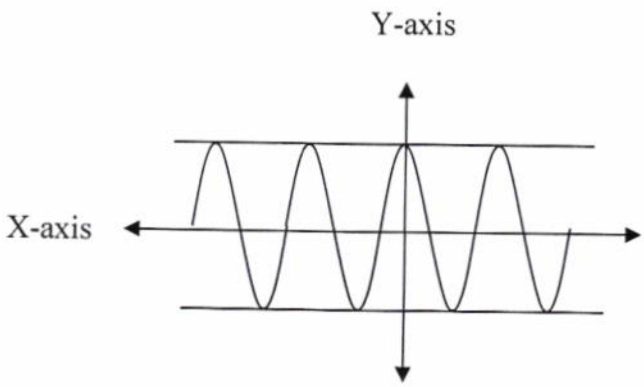
D_F: R

$R_F = [0, 1]$

Example 17: $y = -2\sin^2 x$

D_F: R

$R_F = [-2, 0]$



Example 18: $y = \cos x$

D_F: R

$R_F = [-1, 1]$

Even and Odd Functions: الدوال الزوجية والدوال الفردية

A function (f) is called:

- Even if $\Rightarrow$ $f(-x) = + f(x)$
- Odd if $\Rightarrow$ $f(-x) = - f(x)$

Example 1: $\mathbf{y = x^2}$

$$f(-x) = (-x)^2 = x^2 = + f(x) \Rightarrow \therefore \text{even function}$$

Example 2: $\mathbf{y = x^3}$

$$f(-x) = (-x)^3 = -x^3 = - f(x) \Rightarrow \therefore \text{odd function}$$

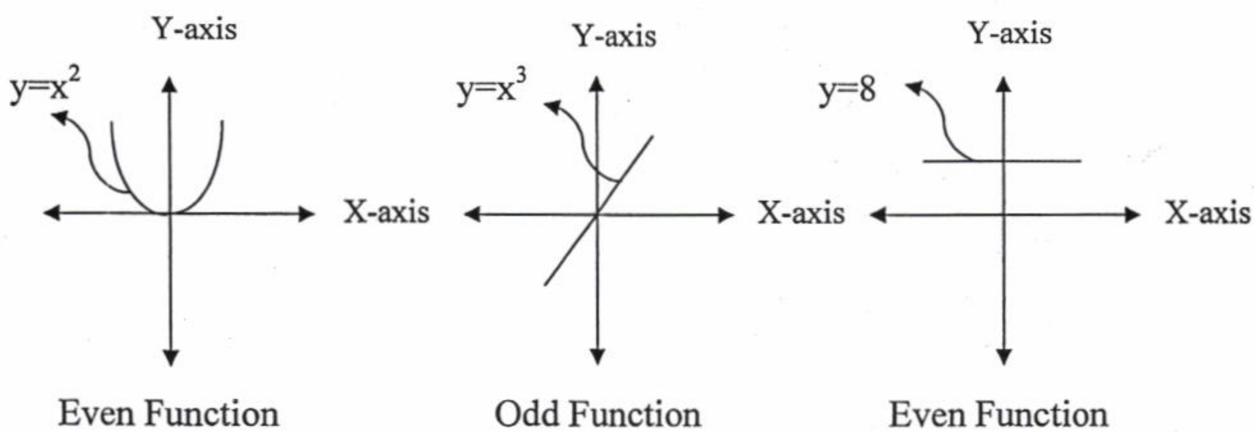
Example 3: $\mathbf{y = 8}$

$$f(-x) = 8 \Rightarrow \therefore \text{even function}$$

Note: For all $x \in D_F$:

- Even if $\Rightarrow$ the function is symmetric about the y-axis.
- Odd if $\Rightarrow$ the function is symmetric about the origin.

Example 4:



Note:

- Odd ± Odd = Odd & Even ± Even = Even
- Odd * Odd = Even & Even * Even = Even
- Odd / Even = Odd = Even / Odd
- Odd * Even = Odd = Even * Odd

Example 5: **y = Sinx**

Odd function (symmetric about the origin)

Example 6: **y = Cosx**

Even function (symmetric about the y-axis)

Example 7: **y = tanx**

$$y = \frac{Sinx}{Cosx} = \frac{Odd}{Even} = \text{Odd function}$$

Example 8: $f(x) = \frac{x2 + x4}{x + Sinx}$

$$f(x) = \frac{Even + Even}{Odd + Odd} = \frac{Even}{Odd} = \text{Odd function}$$

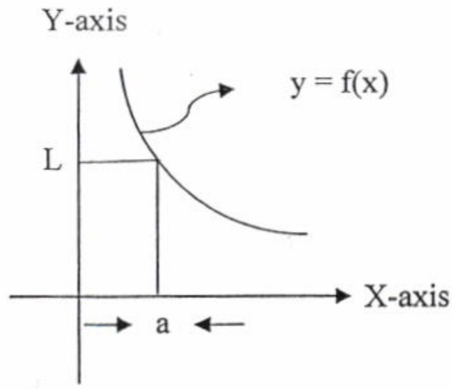
Example 9: $f(x) = x^3 - 2$

$$f(x) = Odd - Even = \text{neither Even nor Odd}$$

Example 10: $f(x) = \frac{x3 + x5}{Sinx + 2}$

$$f(x) = \frac{Odd + Odd}{Odd + Even} = \text{neither Even nor Odd}$$

Limits of a Function: الغايات



Definitions:

$\lim\limits_{x \to a} f(x) = L$ Mean that when a value of (x) close to (a) f(x) approaches the limiting value (L).

$\lim\limits_{x \to a+} f(x) = L$ Mean that (x) approaches (a) from the right.

$\lim\limits_{x \to a-} f(x) = L$ Mean that (x) approaches (a) from the left.

Note: If $\lim\limits_{x \to a+} f(x) = L = \lim\limits_{x \to a-} f(x) = L$ we say that $\lim\limits_{x \to a} f(x) = L$ exist, otherwise the limit doesn't exist.

Example 1: **Find** $\lim\limits_{x \to 1} f(x)$ **where** $f(x) = \begin{cases} x^2 + 1 & \text{when } x \le 1 \\ 3 - x & \text{when } x > 1 \end{cases}$

$$\lim_{x \to 1-} f(x) = x^2 + 1$$

$$= (1)^2 + 1 = 2$$

$$\lim_{x \to 1+} f(x) = 3 - x$$

$$= 3 - 1 = 2$$

$$\because \lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 2 \Rightarrow \therefore \lim_{x \to 1} f(x) \text{ exist}$$

Example 2: **Find** $\lim\limits_{x \to 0} f(x)$ **where** $f(x) = \begin{cases} x^2 + 1 & \text{when } x \ge 0 \\ x & \text{when } x < 0 \end{cases}$

$$\lim_{x \to 0+} f(x) = x^2 + 1 = 0 + 1 = 1$$

$$\lim_{x \to 0-} f(x) = x = 0$$

$$\because \lim_{x \to 0+} f(x) \ne \lim_{x \to 0-} f(x) \Rightarrow \therefore \lim_{x \to 0} f(x) \text{ doesn't exist}$$

Properties of Limits: خصائص الغايات

Let: $\lim_{x\to a} f(x) = L1$

$\lim_{x\to a} g(x) = L2$

& K is a constant, then:

1) $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L1 + L2$

2) $\lim_{x\to a} [f(x) * g(x)] = \lim_{x\to a} f(x) * \lim_{x\to a} g(x)$

3) $\lim_{x\to a} [\frac{f(x)}{g(x)}] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ but $\lim_{x\to a} g(x) \neq 0$

4) $\lim_{x\to a} K * f(x) = K * \lim_{x\to a} f(x)$

5) $\lim_{x\to a} K = K$

6) $\lim_{x\to a} x = a$

7) $\lim_{x\to\infty} \frac{1}{x} = 0$ & $\lim_{x\to\infty} \sqrt{x} = \infty$

8) $\lim_{x\to 0+} \frac{1}{x} = +\infty$ & $\lim_{x\to 0-} \frac{1}{x} = -\infty$ but $\lim_{x\to 0} \frac{x}{1} = 0$

9) $\lim_{x\to 0} sinx = 0$ & $\lim_{x\to 0} cosx = 1$ & $\lim_{x\to 0} tanx = 0$

10) $\lim_{x\to 0} \frac{sinx}{x} = 1$ & $\lim_{x\to 0} \frac{x}{sinx} = 1$

11) $\lim_{x\to 0} \frac{tanx}{x} = 1$ & $\lim_{x\to 0} \frac{x}{tanx} = 1$

12) $\lim_{x\to a} \sin\left(\frac{x^2}{\pi+x}\right) = \sin(\lim_{x\to a} \frac{x^2}{\pi+x})$ Note: sin or cos or any trigonometric functions is the same

13) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ & $\lim_{x\to a} \frac{1}{x^n} = [\lim_{x\to a} \frac{1}{x}]^n$

Note:

Undefined expression in limits:

$\frac{0}{0}, \frac{\infty}{\infty}, \frac{0}{\infty}, \frac{\infty}{0}, 0 * \infty, \infty * \infty, \infty - \infty$ *but we can say* $\infty + \infty = \infty$

أساليب الحل الممكن اتباعها في حل أسئلة الغايات:

1) التعويض المباشر اذا كان الناتج معرف.

2) باستخدام طرق التحليل المختلفة أو المرافق اذا كانت ليست دوال مثلثية.

3) باستخدام الخصائص من 9 الى 13 اذا كانت دوال مثلثية.

4) اذا كانت $x \to \infty$ فهناك ثلاث طرق للحل:

أولا: اذا كانت دوال مثلثية نحول x الى متغير أخر وليكن $\frac{1}{y}$ مثلا وعندما $x \to \infty$ فأن $y \to 0$

ثانيا: اذا كانت دوال كسرية نقسم على أكبر اس موجود في المقام.

ثالثا: اذا كانت دوال غير كسرية وليست مثلثية نحولها الى دوال كسرية بالضرب في المرافق ثم نقسم على أكبر اس موجود في المقام.

5) باستخدام طريقتين أو أكثر.

Evaluate the following limits:

Example 1: $\lim\limits_{x\to1} \frac{x^2+1}{x}$

$= \frac{1+1}{1} = 2$

Example 2: $\lim\limits_{x\to-1} \frac{x^2-1}{x+1}$

$= \lim\limits_{x\to-1} \frac{(x-1)(x+1)}{(x+1)} = -1-1 = -2$

Example 3: $\lim\limits_{x\to1} \frac{x^3-1}{x-1}$

$= \lim\limits_{x\to1} \frac{(x-1)(x^2+x+1)}{(x-1)} = 1+1+1 = 3$

Example 4: $\lim\limits_{x\to0} \frac{x}{1-\sqrt{1-x}}$

$= \lim\limits_{x\to0} \frac{x}{1-\sqrt{1-x}} * \frac{1+\sqrt{1-x}}{1+\sqrt{1-x}}$

$= \lim\limits_{x\to0} \frac{x(1+\sqrt{1-x})}{1-(1-x)} = \lim\limits_{x\to0} \frac{x(1+\sqrt{1-x})}{1-1+x} = 1+1 = 2$

Example 5: $\lim_{x\to 0} \frac{sin3x}{x}$

$= \lim_{x\to 0} \frac{sin3x}{x} * \frac{3}{3} = 1 * 3 = 3$

Example 6: $\lim_{x\to 0} \frac{sin^2 3x}{x^2}$

$= \lim_{x\to 0} \frac{sin^2 3x}{x^2} * \frac{3*3}{3*3} = 9 * 1 * 1 = 9$

Example 7: $\lim_{x\to 1} \frac{sin3x}{sin5x}$

$= \lim_{x\to 1} \frac{sin3x * \frac{3x}{3x}}{sin5x * \frac{5x}{5x}} = \frac{3}{5}$

Example 8: $\lim_{x\to 0} \frac{1-cosx}{x}$

$= \lim_{x\to 0} \frac{1-cosx}{x} * \frac{1+cosx}{1+cosx} = \lim_{x\to 0} \frac{1-cos^2 x}{x(1+cosx)}$

$= \lim_{x\to 0} \frac{sin^2 x}{x(1+cosx)} * \frac{x}{x}$

$= \frac{0}{1+1} = 0$

Example 9: $\lim_{x\to \pi} (sin\frac{x^2}{\pi+x})$

$= sin \lim_{x\to \pi} (\frac{x^2}{\pi+x}) = sin\ \frac{\pi}{2} = 1$

Example 10: $\lim_{x\to \frac{\pi}{2}} (sin\frac{cosx}{\frac{\pi}{2}-x})$

assume $y = \frac{\pi}{2} - x \Rightarrow x = \frac{\pi}{2} - y$ and $x \to \frac{\pi}{2} \Rightarrow y \to 0$

$= \lim_{y\to 0} (sin\frac{\cos(\frac{\pi}{2}-y)}{y}) = sin\lim_{y\to 0} \frac{cos\frac{\pi}{2} cosy + sin\frac{\pi}{2} siny}{y}$

$= sin\lim_{y\to 0} \frac{siny}{y} = \sin 1 = 0.017$

Example 11: $\lim\limits_{x \to \pi} \frac{sinx}{x-\pi}$

assume $y = x - \pi \Rightarrow x = y + \pi$ and $x \to \pi \Rightarrow y \to 0$

$= \lim\limits_{y \to 0} \frac{\sin(y+\pi)}{y} = \lim\limits_{y \to 0} \frac{\sin y \cos\pi + \cos y \sin\pi}{y} = \lim\limits_{y \to 0} \frac{-\sin y}{y} = -1$

Example 12: $\lim\limits_{x \to 2} \frac{cos\frac{\pi}{x}}{x-2}$

$= \lim\limits_{x \to 2} \frac{\sin(\frac{\pi}{2} - \frac{\pi}{x})}{x-2}$

assume $y = x - 2 \Rightarrow x = y + 2$ and $x \to 2 \Rightarrow y \to 0$

$= \lim\limits_{y \to 0} \frac{\sin(\frac{\pi}{2} - \frac{\pi}{y+2})}{y} = \lim\limits_{y \to 0} \frac{\sin(\frac{y\pi + 2\pi - 2\pi}{2(y+2)})}{y}$

$= \lim\limits_{y \to 0} \frac{\sin(\frac{y\pi}{2(y+2)})}{y} * \frac{\frac{\pi}{2(y+2)}}{\frac{\pi}{2(y+2)}} = \lim\limits_{y \to 0} \frac{\pi}{2(y+2)} = \frac{\pi}{4}$

Example 13: $\lim\limits_{x \to \infty} x\, sin(\frac{2}{x})$

assume $x = \frac{1}{y} \Rightarrow y = \frac{1}{x}$ and $x \to \infty \Rightarrow y \to 0$

$= \lim\limits_{y \to 0} \frac{1}{y} \sin 2y$

$= \lim\limits_{y \to 0} \frac{sin2y}{y} * \frac{2}{2} = 2$

Example 14: $\lim\limits_{x \to \infty} \frac{x + sinx}{x + cosx}$

assume $x = \frac{1}{y} \Rightarrow y = \frac{1}{x}$ and $x \to \infty \Rightarrow y \to 0$

$= \lim\limits_{y \to 0} \frac{\frac{1}{y} + \sin(\frac{1}{y})}{\frac{1}{y} + \cos(\frac{1}{y})} = \lim\limits_{y \to 0} \frac{\frac{1 + y\sin(\frac{1}{y})}{y}}{\frac{1 + y\cos(\frac{1}{y})}{y}}$

$= \lim\limits_{y \to 0} \frac{1 + y\sin(\frac{1}{y})}{1 + y\cos(\frac{1}{y})} = \lim\limits_{y \to 0} \frac{1 + \frac{\sin(\frac{1}{y})}{(\frac{1}{y})}}{1 + y\cos(\frac{1}{y})} = 1 + 1 = 2$

Example 15: $\lim_{x\to\infty} \frac{4x^3-2x^2+1}{3x^3-5}$

$= \lim_{x\to\infty} \frac{\frac{4x^3}{x^3}-\frac{2x^2}{x^3}+\frac{1}{x^3}}{\frac{3x^3}{x^3}-\frac{5}{x^3}} = \frac{4}{3}$

Example 16: $\lim_{x\to\infty} (\sqrt{x^2+2x}-x)$

$= \lim_{x\to\infty} \sqrt{x^2+2x}-x * \frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}$

$= \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$

$= \lim_{x\to\infty} \frac{\frac{2x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}+\frac{x}{x}} = \frac{2}{2} = 1$

Example 17: $\lim_{x\to\infty} [cos\left(\frac{\pi x^2+1}{x^2+3}\right)]$

$= cos \lim_{x\to\infty} \left(\frac{\pi x^2+1}{x^2+3}\right)$

$= cos \lim_{x\to\infty} \left(\frac{\frac{\pi x^2}{x^2}+\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{3}{x^2}}\right) = cos\,\pi = -1$

l'Hopital's Rule

قاعدة (أو اسلوب) سنستخدم للتعامل مع الغاية (Limit) التي تكون لها صيغة $\left[\frac{0}{0}\right]$ و $\left[\frac{\infty}{\infty}\right]$

مثل :

① $\lim_{x\to 0} \frac{\sin x}{x} = \left[\frac{0}{0}\right]$

② $\lim_{x\to 2} \frac{x^2-4}{x-2} = \left[\frac{0}{0}\right]$

③ $\lim_{x\to 0} \frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2} = \left[\frac{0}{0}\right]$

④ $\lim_{x\to 0} \frac{x-\sin x}{x^3} = \left[\frac{0}{0}\right]$

⑤ $\lim_{x\to \infty} \frac{x-2x^2}{3x^2+5x} = \left[\frac{\infty}{\infty}\right]$ $\qquad \frac{1-4x}{6x+5} \qquad \frac{-4}{6}$

اسلوب لاستخدام [illegible] :-

① اذا كانت الغاية ذات صيغة $\left[\frac{0}{0}\right]$ أو $\left[\frac{\infty}{\infty}\right]$ أو يمكن تحويلها الى احدى هاتين الصيغتين فنبدأ باشتقاق البسط لوحده والمقام لوحده .

② بعد اكمال الاشتقاق نعوض في الدالة فاذا كان الناتج :-

أ - لا يساوي $\left[\frac{0}{0}\right]$ فنتوقف ونكتب الناتج الذي حصلنا عليه

ب - يساوي $\left[\frac{0}{0}\right]$ / $\left[\frac{\infty}{\infty}\right]$ فنكمل دورة اخرى من الاشتقاق للبسط والمقام

③ بعد اكمال الاشتقاق الثاني نعوض في الدالة ونتحقق من النتائج كما في الخطوة ② اعلاه الى ان نصل الى ناتج لا يساوي $\left[\frac{\text{صفر}}{\text{صفر}}\right]$.

Ex $\lim_{x\to 0} \frac{x-\sin x}{x^3}$ $\qquad$ بالتعويض $= \frac{0}{0}$

$= \frac{1-\cos x}{3x^2}$ $\qquad$ بالتعويض $= \frac{0}{0}$

$= \frac{\sin x}{6x}$ $\qquad$ بالتعويض $= \frac{0}{0}$

$= \frac{\cos x}{6}$

$= \frac{1}{6}$

Ex $\lim_{x\to 0} \frac{1-\cos x}{x+x^2}$ $\qquad$ بالتعويض $= \frac{0}{0}$

$= \frac{\sin x}{1+2x}$ $\qquad$ بالتعويض $\neq \frac{0}{0}$ ولكن يساوي

$= \frac{0}{1} = 0$

Ex $\lim_{x\to 0} \frac{\sin x}{x^2}$ $\qquad$ بالتعويض $= \frac{0}{0}$

$= \frac{\cos x}{2x} \neq \frac{0}{0}$

$= \frac{1}{0} = \infty$ $\qquad$ نكتب الناتج

④ * لا يجوز الحل بهذه الطريقة الا اذا كان منطوق السؤال يشير الى ذلك .

l'Hopital's rule to evaluate the limit

Continuity of a Function:

Continuity of a moving particle on a single path without unbroken curve, gaps, jumps, or holes such curve can be said to be as continuous.

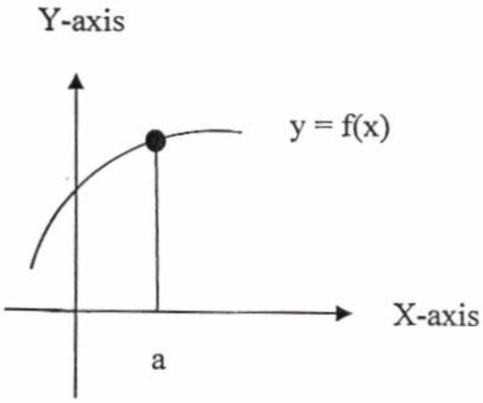


Figure (1) continuous function

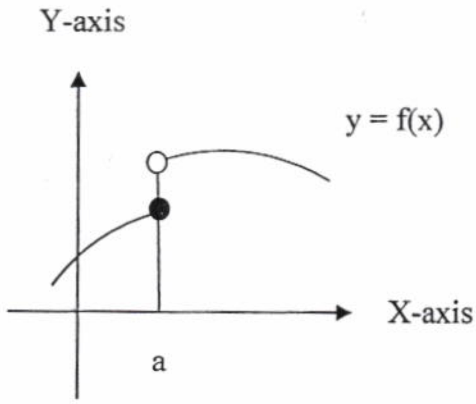


Figure (2) not continuous function

A function is said to be continuous at x= a if the following conditions are satisfied:

1) The $f(a)$ is exist or defined.
2) $\lim_{x \to a} f(x)$ exist.
3) $\lim_{x \to a} f(x) = f(a)$

Otherwise the function is not continuous.

<u>Example 1:</u> Check if the function is continuous at x = 5, & x = 0

$$where\ f(x) = \begin{cases} x^2 - 1 & when\ x \geq 5 \\ x & when\ \ x < 5 \end{cases}$$

At x = 5

$f(5) = (5)^2 - 1 = 24$

$\lim_{x \to 5^+} x^2 - 1 \Rightarrow \lim_{x \to 5^+} 25 - 1 = 24$

$\lim_{x \to 5^-} x \Rightarrow \lim_{x \to 5^-} 5 = 5$

$\lim_{x \to 5^+} f(x) \neq \lim_{x \to 5^-} f(x)$

The limit does not exist, therefore the function not continuous at x = 5

At x = 0

f(0) = does not exist, therefore the function not continuous at x = 0

Example 2: **Find the constant (a) and (b) if the function is:**

$$f(x) = \begin{cases} x^2 + a & \text{when } x \geq 0 \\ 3 + b & \text{when } -1 \leq x < 0 \\ x + 5 & \text{when } x < -1 \end{cases}$$

And the function is continuous at x = 0 and x = -1

At x = 0

$f(0) = (0)^2 - a = a$

$\lim_{x \to 0^+} x^2 + a = a$

$\lim_{x \to 0^-} 3 + b = 3 + b$

The limit must be exist so a = 3 + b

Then the function equal the limit value a = 3 + b ……… (1)

At x= -1

f(-1) = 3 +b

$\lim_{x \to -1^+} 3 + b = 3 + b$

$\lim_{x \to -1^-} x + 5 = 4$

The limit must be exist so 4 = 3 + b then b = 1

Then the function equal the limit value 4 =3+b then b = 1 substitute in equation (1) then a = 3+1 = 4

<u>Example 2:</u> For $x \neq 2$ the function is equal to $\frac{x^2+3x-10}{x-2}$, find the value of f(2) to make the function continuous at x = 2

The limit to be exist must be equal from the left and the right so:

$$\lim_{x\to 2} \frac{x^2+3x-10}{x-2} \Rightarrow \lim_{x\to 2} \frac{(x+5)(x-2)}{(x-2)}$$

$$\lim_{x\to 2} (x+5) = 7$$

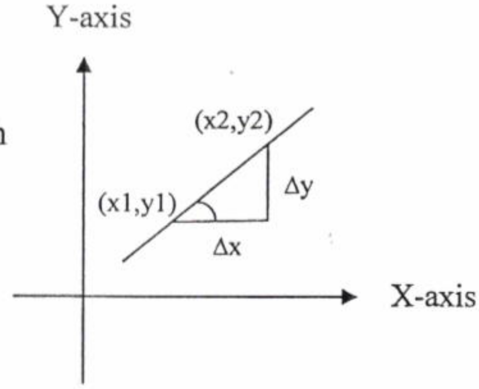
To be continuous then:

$$f(x) = \lim_{x\to 2} f(x) = 7$$

Equation of Straight Lines and Circles:

A. Equation of Straight Line:

- The slope (m) of a straight line passing through Points (x1, y1) & (x2, y2) is:

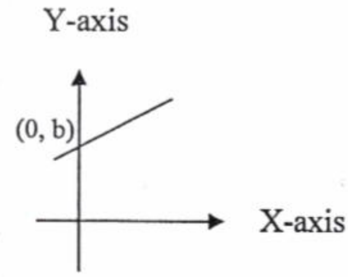


$$m = \frac{\Delta y}{\Delta x} = \frac{y2 - y1}{x2 - x1} \;\; or \;\; \frac{y1 - y2}{x1 - x2}$$

- The equation of a straight line passing through (x1, y1) and has a slop (m) is:

$y - y1 = m(x - x1)$ ……… (The point - slop equation of the line)

- The general formula for the equation of A straight line with a slope (m)



$y - y1 = m(x - x1)$

$y - b = m(x - 0)$

$y = mx + b$ ……… (The point - intercept equation of the line)

<u>**Note 1:**</u> Two lines are parallel if and only if they have the same slopes.

$$L1\ parallel\ L2\ \ if\ \ m1\ =\ m2$$

<u>**Note 2:**</u> Two lines are perpendicular if and only if they have the product of their slopes is (-1).

$$L1\ perpendicular\ L2\ if\ m1\ \times\ m2\ = -1 \Rightarrow m1 = \frac{-1}{m2}$$

Example 1: **Find the equation of the line passing through (-2, -1) & (3, 4) ?**

$$m = \frac{y2 - y1}{x2 - x1} = \frac{4 + 1}{3 + 2} = 1$$

Using (-2, -1) we find:

$$y - y1 = m(x - x1)$$

$$y + 1 = 1(x + 2)$$

$$y = x + 1$$

Or using (3, 4) we find:

$$y - 4 = 1(x - 3)$$

$$y = x + 1$$

Example 2: **Find the slope and y-intercept for $8x + 5y = 20$**

$$8x + 5y = 20$$

$$5y = -8x + 20$$

$$y = \frac{-8}{5}x + 4$$

$$m = \frac{-8}{5} \ \& \ b = 4$$

Example 3: **Find the equation of the line passing through the origin and the point of intersection of $L1 \Rightarrow x + y = 2$ & $L2 \Rightarrow 2x - y = -5$?**

$$x + y = 2 \Rightarrow x = 2 - y$$

$$2(2 - y) - y = -5$$

$$y = 3 \Rightarrow x = -1$$

The point of intersection (-1, 3)

$$m = \frac{y2 - y1}{x2 - x1} = \frac{3 + 0}{-1 + 0} = -3$$

Using (0, 0) or (-1, 3) we find:

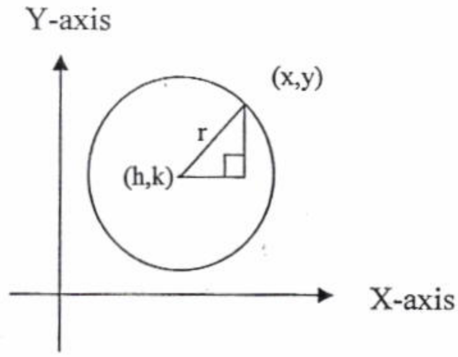
$$y - y1 = m(x - x1)$$

$$y - 0 = -3(x - 0) \Rightarrow y + 3x = 0$$

B. Equation of Circle:

The equation of a circle centered at (h, k) and has a radius (r) is:

$$(x-h)^2+(y-k)^2=r^2$$



<u>Example 1:</u> Find the radius and coordinate of center for:

$$\boldsymbol{x^2+y^2+4x-2y+1=0}$$

$$x^2+4x+4-4+y^2-2y+1=0$$

$$(x+2)(x+2)-4+(y-1)(y-1)=0$$

$$(x+2)^2+(y-1)^2=4$$

The coordinate of center is (-2, 1) and the radius is (2) unit length

Or $\quad x^2+y^2+4x-2y+1=0 \quad \Rightarrow eq: x^2+y^2+ax+by+c=0$

$$h=\frac{-(4)}{2}=-2 \qquad \Rightarrow h=\frac{-(a)}{2}$$

$$k=\frac{-(-2)}{2}=1 \qquad \Rightarrow k=\frac{-(b)}{2}$$

$$r=\sqrt{(-2)^2+(1)^2-1}=2 \qquad \Rightarrow r=\sqrt{h^2+k^2-c}$$

<u>Example 2:</u> Find the equation of the circle centered at (1, -2) and passing through the point (7, 4)?

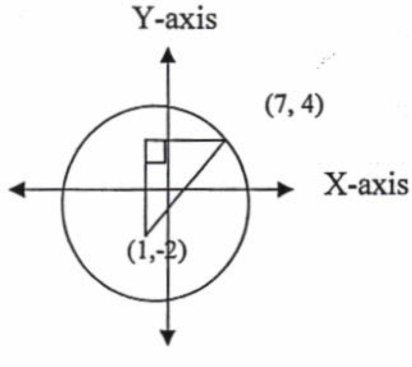
$$d=r=\sqrt{(\Delta x)^2+(\Delta y)^2}$$

$$=\sqrt{(7-1)^2+(4+2)^2}$$

$$=\sqrt{72}\ unit\ length$$

The equation is:

$$(x-1)^2+(y+2)^2=72$$



Example 3: **For this equation: $3y^2 - 12y + 3x^2 + 6x = 18$ find:**

- **The center and the radius of the circle**
- **The equation of the circle**
- **The area of the circle**

$3y^2 - 12y + 3x^2 + 6x = 18$

$[3x^2 + 3y^2 + 6x - 12y - 18 = 0] \div 3$

$x^2 + y^2 + 2x - 4y - 6 = 0$

$$h = \frac{-(a)}{2} = \frac{-(2)}{2} = -1 \;\&\; k = \frac{-(b)}{2} = \frac{-(-4)}{2} = 2$$

The point of the center is: (-1, 2)

$r = \sqrt{h^2 + k^2 - c} = \sqrt{(-1)^2 + (2)^2 + 6} = \sqrt{11}$ *unit length*

The equation of the circle is:

$(x + 1)^2 + (y - 2)^2 = 11$

The area of circle is:

$A = \pi r^2 = 3.14 * 11 = 34.54$ *unit area*

Equations of Straight line and circle

A.) Straight line

* The Slope (m) of a Straight line passing through the points (x_1, y_1), (x_2, y_2) is $m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$
* The equation of a Straight line passing through (x_0, y_0) and has a Slope (m) is $y - y_0 = m(x - x_0)$
* The general formula for the equation of a Straight line with a Slope (m) and y-intercept is $y = mx + b$

Ex.) Find the Slope (m) and y-intercept (b)

① $8x + 5y = 20$

$5y = -8x + 20$

$y = -\frac{8}{5}x + \frac{20}{5}$

$y = mx + b$

$\therefore m = -\frac{8}{5}$

$b = \frac{20}{5} = 4$

② $x - 2y = 4$

$2y = x - 4$

$y = \frac{1}{2}x - \frac{4}{2}$

$m = \frac{1}{2}$

$b = -\frac{4}{2} = -2$

Ex.) Find the equation of the line passing through the origin and the point of intersection of

$L_1: x + y = 2$ and $L_2: 2x - y = -5$

$x + y = 2$

$2x - y = -5$

$3x = -3 \Rightarrow x = -1 \Rightarrow y = 3 \Rightarrow (-1, 3)$

$m = \frac{\Delta y}{\Delta x} = \frac{3 - 0}{-1 - 0} = -3$

$y - y_0 = m(x - x_0) \Rightarrow y - 0 = -3(x - 0)$

Notes: * two lines are parallel if and if they have the same slope $L_1 \parallel L_2$ if $m_1 = m_2$

* two lines are perpendicular if and only if (orthogonal, vertical) the product of their Slopes is -1

$L_1 \perp L_2$ if $m_1 * m_2 = -1$ or $m_1 = -\frac{1}{m_2}$

B) Circle: The equation of the circle with a centre (h, k) and has a radius (r) is

$(x-h)^2 + (y-k)^2 = r^2$

$(x-2)^2 + (y+3)^2 = 9$ اوجد المركز ونصف القطر

المركز $(2,-3)$ / نق $= 3$

$(x+2)^2 + (y+3)^2 = 16$

المركز $(-2,-3)$ / نق $= 4$

Ex.) Find the radius and coordinate of the centre

$x^2 + y^2 + 4x - 2y + 1 = 0$

$x^2 + 4x + y^2 - 2y = -1$

$x^2 + 4x + 4 - 4 + y^2 - 2y + 1 - 1 = -1$ نضيف ونطرح مربع ½ معامل x

$(x+2)^2 - 4 + (y-1)^2 - 1 = -1$ المركز نق

$(x+2)^2 + (y-1)^2 = 4 \implies (-2, 1) \;,\; 2$

Ex.) For what value of K does the circle $(x-K)^2 + (y-2K)^2 = 10$ pass through the point $(1,1)$

$(1-K)^2 + (1-2K)^2 = 10$

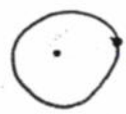
$1 - 2K + K^2 + 1 - 4K + 4K^2 - 10 = 0$

$5K^2 - 6K - 8 = 0$

$(5K+4)(K-2) = 0$ either $K = -\frac{4}{5}$ or $K = 2$

Ex.) Find the equation of the circle centered at $(1,-2)$ and passing through $(7,4)$

$r =$ المسافة بين نقطتين $= \sqrt{(7-1)^2 + (4-(-2))^2} = \sqrt{72}$



the equation is $(x-1)^2 + (y+2)^2 = 72$

H.W * Find the equation of the circle that passes through the points $A(2,3)$, $B(-4,3)$ and $C(3,2)$.

* Find the equation of the circle which passes through $(10,2)$, $(9,-3)$ and the centre of the circle lies on the y-axis?

تقع

هنا يعني ان احداثيات المركز $(0, K)$

Differentiation

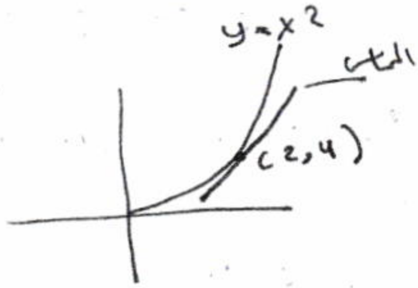
we call the derivative of the function $f(x)$ as

$$f'(x),\ \frac{d f(x)}{dx} \text{ or } \frac{dy}{dx} \text{ or } y'$$

* The derivative of a function at a point $x=a$ is the Slope of the tangent line to the curve.

عند نقطة التماس ميل المماس = مشتقة الدالة

Ex.) Find the equation of the tangent line to the curve $y = x^2$ at $(2, 4)$.



$$\frac{dy}{dx} = 2x$$

$$m = \frac{dy}{dx} = 2 * 2 = 4 \quad \text{at } (2,4)$$

نعوض معادلة المماس $(2, 4)$

$$y - y_0 = m(x - x_0)$$

$$y - 4 = 4(x - 2)$$ معادلة المماس عند $(2,4)$

properties of the derivative

1) $\frac{d}{dx}(c) = 0$ where c is a constant
2) $\frac{d}{dx}[f(x) \mp g(x)] = f'(x) \mp g'(x)$
3) $\frac{d}{dx}\, c f(x) = c f'(x)$
4) $\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$
5) $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x).f'(x) - f(x).g'(x)}{[g(x)]^2}$
6) $\frac{d}{dx}(x^n) = n x^{n-1}$
7) $\frac{d}{dx}(\sin x) = \cos x$
8) $\frac{d}{dx}(\cos x) = -\sin x$
9) $\frac{d}{dx}(\tan x) = \sec^2 x$
10) $\frac{d}{dx}(\cot x) = -\csc^2 x$
11) $\frac{d}{dx}(\sec x) = \sec x \tan x$
12) $\frac{d}{dx}(\csc x) = -\csc x \cot x$

Example: Find $\frac{dy}{dx}$ for the following

* $y = x^2(x^3+2) \Rightarrow y' = x^2(3x^2) + (x^3+2) * 2x$
* $y = \frac{3}{x} \Rightarrow y' = \frac{x * 0 - 3 * 1}{x^2} = -\frac{3}{x^2}$

or $y = 3x^{-1} \Rightarrow y' = -3x^{-2} = -\frac{3}{x^2}$

* $y = \tan^2 x \Rightarrow y' = 2\tan x \cdot \sec^2 x$

* $y = \sin^2 x^3 = (\sin x^3)^2$

$y' = 2\sin x^3 \cos x^3 * 3x^2$

* $y = \sqrt{\sec(3x^3)} \Rightarrow y = (\sec(3x^3))^{1/2}$

$y' = \frac{1}{2}[\sec(3x^3)]^{-1/2} * \sec 3x^3 \tan 3x^3 * 9x^2$

* $y = \sqrt{\sec(x\cos x)} \Rightarrow y = [\sec(x\cos x)]^{1/2}$

$y' = \frac{1}{2}[\sec(x\cos x)]^{-1/2} * \sec(x\cos x)\tan(x\cos x)$

$* [x(-\sin x) + \cos x\,(1)]$

* $y = \tan^2(\cos\frac{1}{x}) = [\tan(\cos\frac{1}{x})]^2$

$y' = 2\tan(\cos\frac{1}{x}) \cdot \sec^2(\cos\frac{1}{x}) * -\sin\frac{1}{x} * -x^{-2}$

Higher order derivative

Find $\frac{d^4y}{dx^4}$ for $y = x^6 - 3x^4 + \cos x$

$y' = 6x^5 - 12x^3 - \sin x$

$y'' = 30x^4 - 36x^2 - \cos x$

$y''' = 120x^3 - 72x + \sin x$

$y'''' = 360x^2 - 72 + \cos x$

$\frac{d^2y}{dx^2} = y'' = \frac{d}{dx}y' = \frac{d}{dx}\frac{dy}{dx}$

$\frac{d^3y}{dx^3} = y''' = \frac{d}{dx}\frac{d^2y}{dx^2}$

Chain Rule:

If $y = f(t)$ and $x = g(t)$ then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

Ex.) if $y = \sin t$, $x = \cos t$ find $\frac{dy}{dx}$

$$\frac{dy}{dt} = \cos t \quad , \quad \frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t}{-\sin t} = -\cot t$$

Ex.) Find $\frac{dy}{dx}$ if $y = t^3$ and $t = x^2 + 2$

$$\frac{dy}{dt} = 3t^2 \quad , \quad \frac{dt}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 3t^2 * 2x = 3(x^2+2)^2 * 2x$$

or $y = (x^2+2)^3$ بالتعويض المباشر

$$\frac{dy}{dx} = 3(x^2+2)^2 * 2x$$

Ex.) Find $\frac{d^2y}{dx^2}$ if $y = (t^2+1)^4$, $x = t^2 + 5$

$$\frac{dy}{dt} = 4(t^2+1)^3 * 2t \quad , \quad \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4(t^2+1)^3 * 2t}{2t} = 4(t^2+1)^3$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{12(t^2+1)^2 * 2t}{2t} = 12(t^2+1)^2$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}}}$$

implicit differentiation الاشتقاق الضمني

we differentiate both sides with respect to x

Ex) Find $\frac{dy}{dx}$ if $x^2 + xy + y - x = 0$

$$2x + xy' + y * 1 + y' - 1 = 0$$
$$2x + y - 1 = -xy' - y'$$
$$2x + y - 1 = y'(-x - 1)$$
$$y' = \frac{2x + y - 1}{-x - 1}$$

Ex.) $\sin y + x \sin x = 1$

$$\cos y * y' + x \cos x + \sin x = 0$$
$$y' \cos y = -x \cos x - \sin x$$
$$y' = \frac{-x \cos x - \sin x}{\cos y}$$

Ex) if $x^2 + y^2 = 1$ Show that $\frac{d^2y}{dx^2} = -\frac{1}{y^3}$

$$2x + 2y * \frac{dy}{dx} = 0 \Longrightarrow 2y \frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$
$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$
$$\frac{d^2y}{dx^2} = -\frac{y + \frac{x^2}{y}}{y^2} = -\frac{\frac{y^2 + x^2}{y}}{y^2}$$
$$= -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3}$$

Ex.) Find the equation of the tangent to the curve $y = \sin\sqrt{x}$ at $(\pi^2, 0)$

$$\frac{dy}{dx} = \cos\sqrt{x} * \frac{1}{2\sqrt{x}} = \cos\pi * \frac{1}{2\pi} = -\frac{1}{2\pi} \text{ at } (\pi^2, 0)$$

$$y - y_0 = m(x - x_0) \Rightarrow y - 0 = -\frac{1}{2\pi}(x - \pi^2)$$

Application of Derivatives :-

increasing function when $f'(x) > 0$ — المشتقة موجبة

Decreasing function when $f'(x) < 0$ — المشتقة سالبة

Horizontal tangent عندما $f'(x) = 0$.

وهذا يعني ان مماس الدالة افقي وان الدالة تتغير من تزايدية الى تناقصية او بالعكس.

Ex.) Graph the function $y = f(x) = x^3 - 3x^2 + 4$

① نجد نقاط التقاطع مع المحاور

when $x = 0 \Rightarrow y = 4 \Rightarrow (0, 4)$

when $y = 0 \Rightarrow 0 = (x+1)(x-2)^2 \Rightarrow x = -1 \Rightarrow (-1, 0)$

$x = 2 \Rightarrow (2, 0)$

② نجد الفترات التي تكون فيها المشتقة الاولى موجبة/سالبة/صفر

$y' = 3x^2 - 6x = 3x(x - 2)$

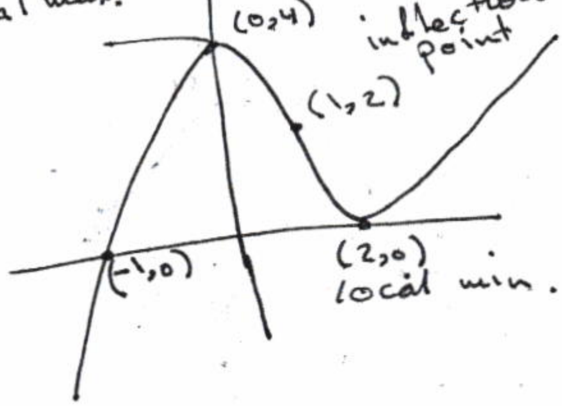
تزايدية — تناقصية — تزايدية

++ (0) −− (2) ++

③ نجد المشتقة الثانية

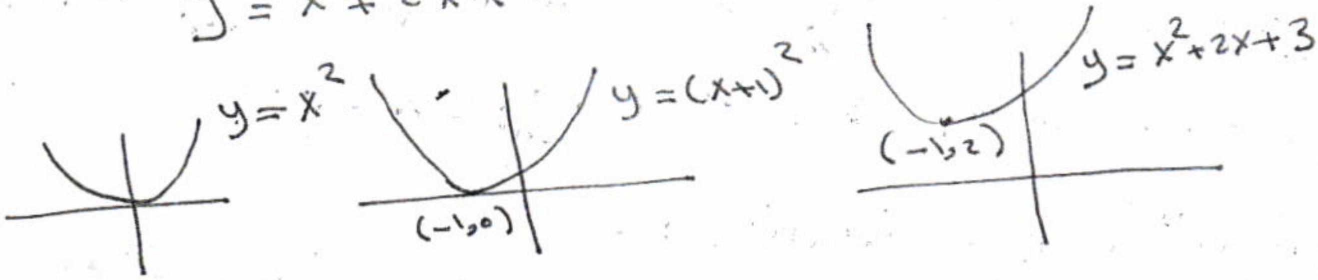
$y'' = 6x - 6 \Rightarrow x = 1$

x	y
-1	0
0	4
1	2
2	0
3	4

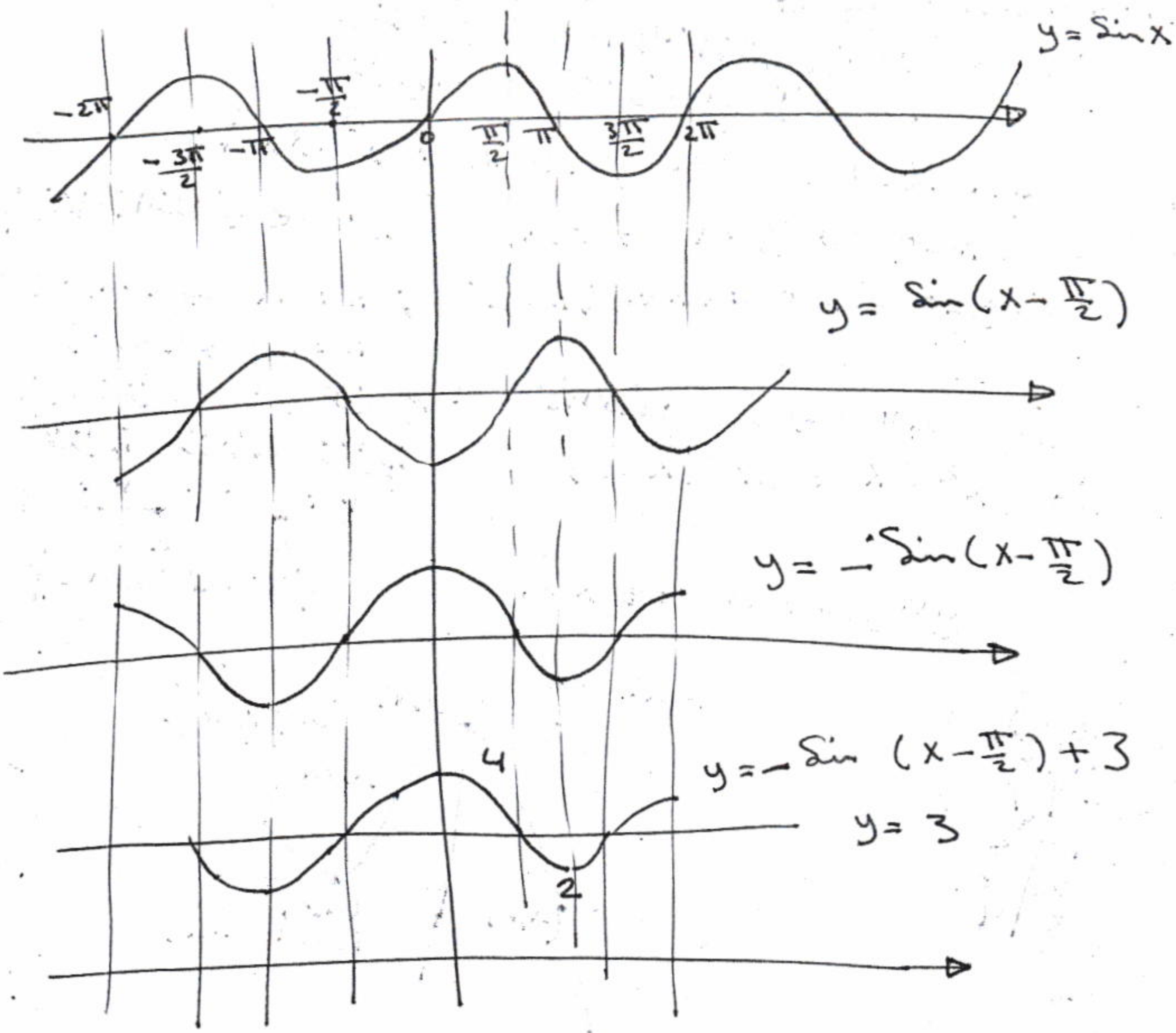


Ex.) Graph $y = x^2 + 2x + 3$

$$y = x^2 + 2x + 1 - 1 + 3 \Rightarrow y = (x+1)^2 + 2$$



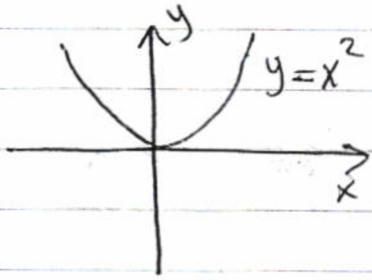
Ex.) Graph $y = -\sin\left(x - \frac{\pi}{2}\right) + 3$

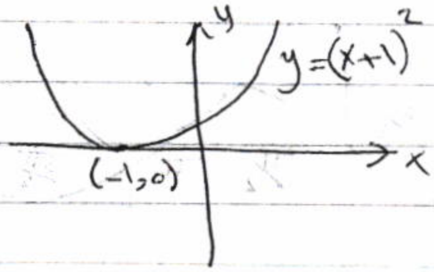


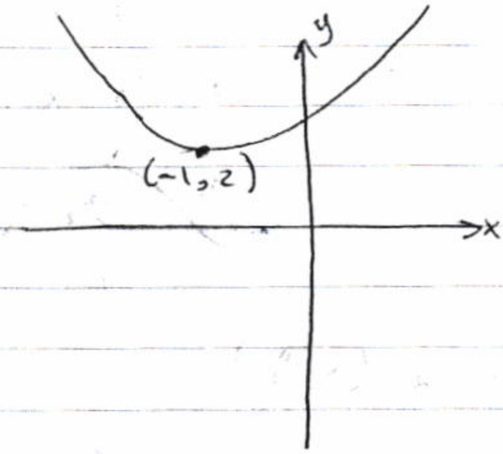
Rf : $2 \leq y \leq 4$

Ex.) Graph $y = x^2 + 2x + 3 = x^2 + 2x + 1 - 1 + 3$

$y = (x+1)^2 + 2$



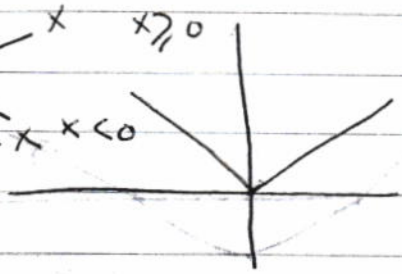


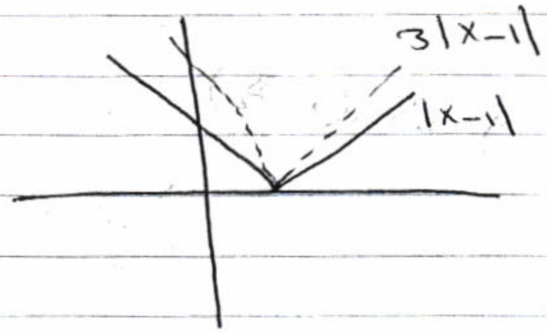


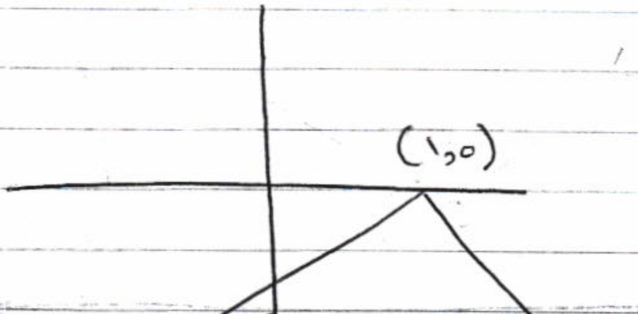
Graph $y = |x-1|$

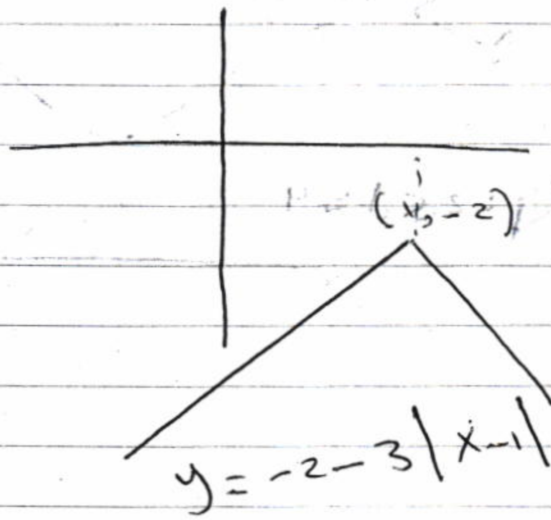
Ex.) $y = -2 - 3|x-1|$

$y = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

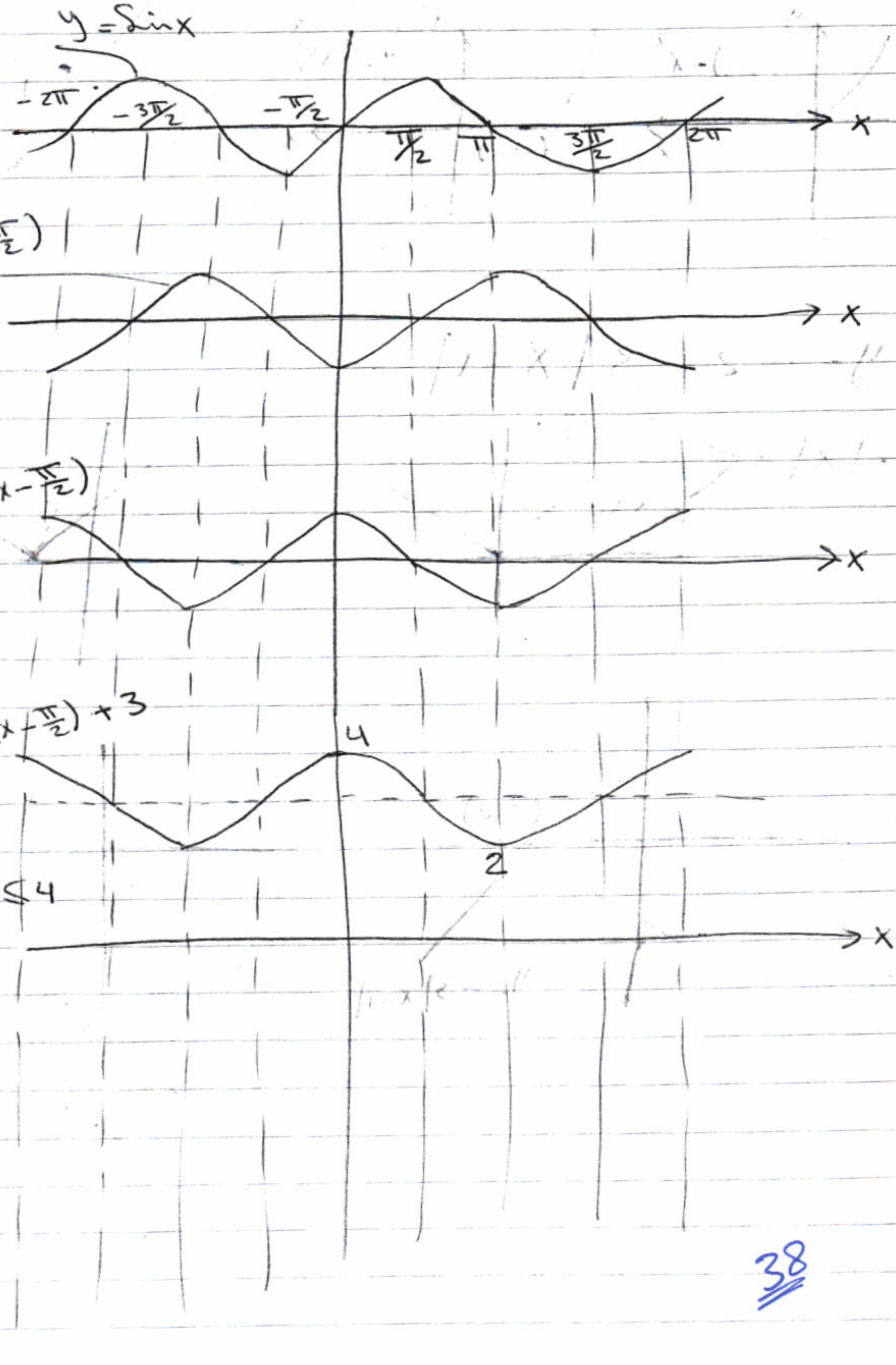








Ex.) Graph $y = -\sin\left(x - \frac{\pi}{2}\right) + 3$



$R_f: 2 \leq y \leq 4$

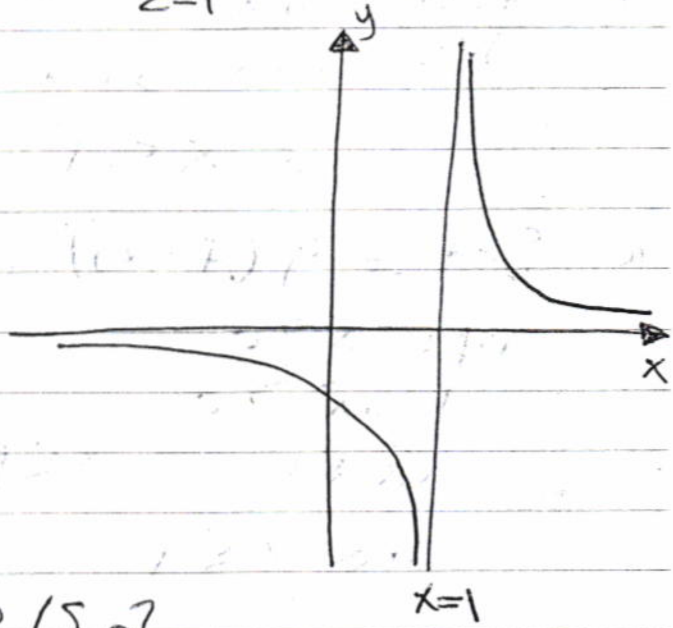
Ex) if $f(x+5) = \dfrac{1}{x+4}$

① Graph $f(x)$ ② find the Domain and Range
③ Evaluate $\lim\limits_{x \to 1} f(x)$

let $z = x+5 \Rightarrow z-5 = x$

$f(z) = \dfrac{1}{(z-5)+4} \Rightarrow f(z) = \dfrac{1}{z-1}$

$f(x) = \dfrac{1}{x-1}$



$D_f: x \neq 1$
$D_f: R/\{1\}$

Range: $y = \dfrac{1}{x-1}$

$yx - y = 1$
$yx = 1 + y$
$x = \dfrac{1+y}{y} \quad y \neq 0 \Rightarrow R_f = R/\{0\}$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} \dfrac{1}{x-1} = \infty \,, \lim\limits_{x \to 1^-} \dfrac{1}{x-1} = -\infty$

$\because \lim\limits_{x \to 1^+} \neq \lim\limits_{x \to 1^-}$

$\therefore \lim\limits_{x \to 1} f(x)$ not exist if $f(x)=\sqrt{x}$, $g(x)=\sqrt{1-x}$ find the Domain of ⓐ $f(x)+g(x)$ ⓑ $\frac{f(x)}{g(x)}$ ⓒ $g \circ f$

الحل

ⓐ $f(x)+g(x)=\sqrt{x}+\sqrt{1-x}$

$x \geq 0$

$1-x \geq 0 \Rightarrow x \leq 1$

$\therefore \; 0 \leq x \leq 1$

ⓑ $\frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{1-x}}$ $\sqrt{x} \Rightarrow x \geq 0$, $\sqrt{1-x} \Rightarrow 1-x > 0 \Rightarrow x < 1$

$0 \leq x < 1$

ⓒ $g \circ f = g(f(x)) = \sqrt{1-\sqrt{x}}$

$D_f: \; x \geq 0$

$1-\sqrt{x} \geq 0$

$\sqrt{x} \leq 1 \Rightarrow x \leq 1$

$0 \leq x \leq 1$

Ex) $f(x)=\sqrt{x}$, $g(x)=x^3+1$

$f \circ g = f(g(x)) = \sqrt{x^3+1}$

$g \circ f = g(f(x)) = (\sqrt{x})^3+1$

Q1

A body moves in a straight line according to the law of motion

$S = t^3 - 4t^2 - 3t$

Find its acceleration at each instant when the velocity is zero.

$v = \frac{ds}{dt} = 3t^2 - 8t - 3 \quad , \quad a = \frac{dv}{dt} = 6t - 8$

$0 = 3t^2 - 8t - 3 \Longrightarrow 0 = (3t+1)(t-3)$

$\therefore \; t = -\frac{1}{3} \text{ or } t = 3$

$\therefore \; a = -10 \text{ or } a = 10$

Q2 Find the velocity $v = \frac{ds}{dt}$ and acceleration $a = \frac{dv}{dt}$:

① $S = 2t^3 - 5t^2 + 4t - 3$

② $S = gt^2/2 + V_0 t + S_0$; $(g, V_0, S_0$ constants$)$

③ $S = (2t+3)^2$

Q3 Find y' and y''

① $y = 2x^4 - 4x^2 - 8$

② $2y = 6x^4 - 18x^2 - 12x$

③ $Y = \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} - x + 3$

④ $Y = (3x-1)(2x+5)$

⑤ $Y = \sin 3x \cdot \tan^2 (x+2)^2 \cdot \sqrt{x-1}$

⑥ $Y = \frac{1}{\sec^2 (x^2 y + h)}$ where h constant

Integration

1. $\int dx = x + c$
2. $\int (f(x) \mp g(x))\, dx = \int f(x)\, dx \mp \int g(x)\, dx$
3. $\int K . f(x)\, dx = K \int f(x)\, dx$ where K is constant
4. $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$
5. $\int \frac{dx}{x} = \ln |x| + c$

Ex.) Evaluate the following integrals

1. $\int (x^2 + x)\, dx = \int x^2 dx + \int x\, dx = \frac{x^3}{3} + \frac{x^2}{2} + c$

2- $\int 4x\, dx = 4 \int x\, dx = 4 \frac{x^2}{2} + c = 2x^2 + c$

3- $\int (x^2+1)^5 . 2x\, dx = \frac{(x^2+1)^6}{6} + c$

4. $\int (x^3 + 3x + 5)(x^2+1)\, dx$

$= \int (x^3 + 3x + 5)\, (x^2+1) \frac{3}{3}\, dx$

$= \frac{1}{3} \frac{(x^3+3x+5)^2}{2} + c$

5. $\int \frac{x^2 + x}{7}\, dx = \frac{1}{7} \int x^2 dx + \frac{1}{7} \int x\, dx$

$= \frac{x^3}{21} + \frac{x^2}{14} + c$

6. $\int \frac{dx}{x^5} = \int x^{-5} dx = \frac{x^{-4}}{-4} + c = -\frac{1}{4x^4} + c$

Integration of trigonometric function

Since $\frac{d}{dx}(\sin x) = \cos x$ then

$$\int \cos x \, dx = \sin x + c$$

Similarly for other trigonometric function

1) $\int \cos u \, du = \sin u + c$

2) $\int \sin u \, du = -\cos u + c$

3) $\int \sec^2 u \, du = \tan u + c$

4) $\int \csc^2 u \, du = -\cot u + c$

5) $\int \sec u \tan u \, du = \sec u + c$

6) $\int \csc u \cot u \, du = -\csc u + c$

7) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + c$

8) $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + c$

9) $\int \sec x \, dx = \ln|\sec x + \tan x| + c$

10) $\int \csc x \, dx = \ln|\csc x - \cot x| + c$

Ex.)

① $\int \cos 3x\, dx = \frac{1}{3}\int \cos 3x * 3dx = \frac{1}{3}\sin 3x + C$

② $\int \sin 7x\, dx = -\frac{1}{7}\cos 7x + C$

③ $\int \sec^2(x+3)\, dx = \tan(x+3) + C$

④ $\int \cot 2x \cdot \csc 2x\, dx = -\frac{1}{2}\int -\csc 2x \cot 2x * 2\, dx$

$= -\frac{1}{2}\csc 2x + C$

⑤ $\int x \sin(2x^2) dx = \frac{1}{4}\int \sin(2x^2) * 4x\, dx$

$= -\frac{1}{4}\cos 2x^2 + C$

⑥ $\int 2\sin x \cos x\, dx = 2\int (\sin x)\cos x\, dx$

$= 2\,\frac{\sin^2 x}{2} + C = \sin^2 x + C$

Definite integrals :-

$\int_a^b f(x)\, dx$

a is called the lower bound

b " " " upper "

Properties of definite integrals :-

1) $\int_a^a f(x)\, dx = 0$

ex) $\int_3^3 \frac{x^3+3x^2-2}{\cos^3 x}\, dx = 0$

2.) $\int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx$

ex.) $\int_{1}^{3} 2x\,dx = -2\int_{3}^{1} x\,dx$

3.) $\int_{a}^{b} f(x)\,dx = \int_{a}^{c} f(x)\,dx + \int_{c}^{b} f(x)\,dx$

ex.) $\int_{1}^{3} 2x\,dx = \int_{1}^{2} 2x\,dx + \int_{2}^{3} 2x\,dx$

4.) $\int_{a}^{b} K f(x)\,dx = K\int_{a}^{b} f(x)\,dx$

5.) $\int_{a}^{b} (f(x) \mp g(x))\,dx = \int_{a}^{b} f(x)\,dx \mp \int_{a}^{b} g(x)\,dx$

6.) if (f) is even function then

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$

ex.) $\int_{-4}^{4} (x^2-5)\,dx = 2\int_{0}^{4} (x^2-5)\,dx$

7.) if (f) is odd function then

$\int_{-a}^{a} f(x)\,dx = 0$

$\int_{-4}^{4} 2x\,dx = 0$